

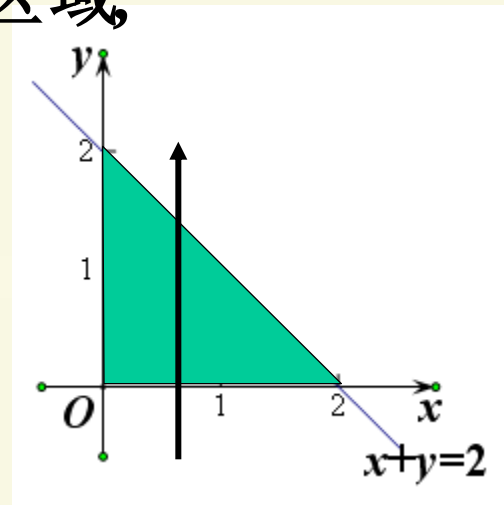
第九章
重积分
同步练习

9.1.一

1. 设 D 由坐标轴与直线 $x + y = 2$ 围成的区域,
则 $\iint_D (3x + 2y) d\sigma = \underline{\hspace{2cm}}$

解 D 是 X -型区域

$$\begin{aligned}\iint_D (3x + 2y) d\sigma &= \int_0^2 dx \int_0^{2-x} (3x + 2y) dy \\&= \int_0^2 [3xy + y^2]_0^{2-x} dx \\&= \int_0^2 [3x(2-x) + (2-x)^2] dx \\&= \int_0^2 (6x - 3x^2 + 2 - 2x + x^2) dx \\&= \int_0^2 (-2x^2 + 4x + 2) dx \\&= \left[-\frac{2}{3}x^3 + 2x^2 + 2x\right]_0^2 \\&= \frac{20}{3}\end{aligned}$$



9.1.一

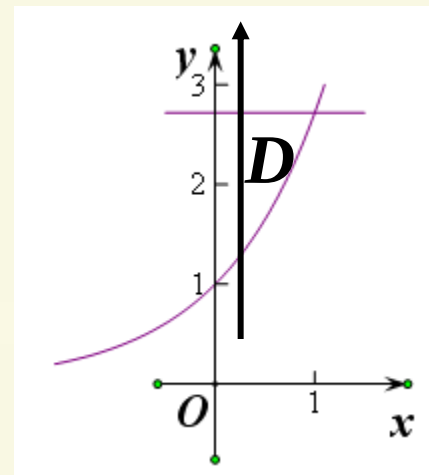
2. 设函数 $f(x, y)$ 连续, 交换积分次序

$$\int_1^e dy \int_0^{\ln y} f(x, y) dx = \underline{\hspace{2cm}}$$

解 D 是 Y -型区域

$$\int_1^e dy \int_0^{\ln y} f(x, y) dx = \iint_D f(x, y) dx dy$$

$$D \text{ 是 } X\text{-型区域} \quad = \int_0^1 dx \int_{e^x}^e f(x, y) dy$$



9.1.一

3. 二次积分 $\int_0^1 dx \int_0^x y dy = \underline{\hspace{2cm}}$

$$\text{解 } \int_0^1 dx \int_0^x y dy = \int_0^1 \left[\frac{1}{2} y^2 \right]_0^x dx = \frac{1}{2} \int_0^1 x^2 dx$$

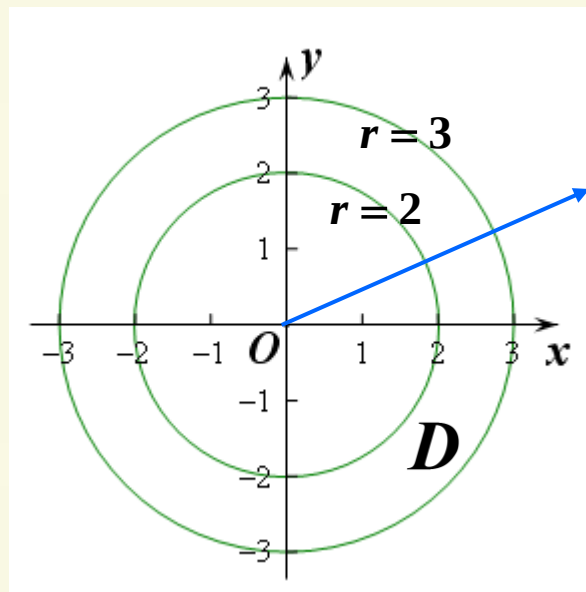
$$= \frac{1}{6} [x^3]_0^1 = \frac{1}{6}$$

9.1.一

4. 设 $D = \{(x, y) \mid 4 \leq x^2 + y^2 \leq 9\}$, 则 $\iint_D d\sigma = \underline{\hspace{2cm}}$

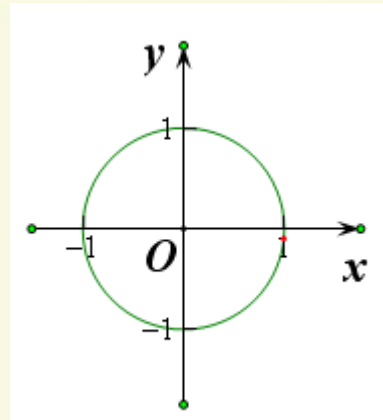
$$\begin{aligned}\text{解 } \iint_D d\sigma &= S_D = 9\pi - 4\pi \\ &= 5\pi\end{aligned}$$

$$\begin{aligned}\iint_D d\sigma &= \iint_D r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_2^3 r dr \\ &= \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_2^3 d\theta \\ &= \frac{5}{2} \int_0^{2\pi} d\theta \\ &= 5\pi\end{aligned}$$



9.1.一

5. 设 $D = \{(x, y) \mid x^2 + y^2 \leq 1, y \geq 0\}$,
则 $\iint_D (x^2 + y^2) d\sigma = \underline{\hspace{2cm}}$



$$\begin{aligned}\text{解 } \iint_D (x^2 + y^2) d\sigma &= \iint_D r^2 \cdot r dr d\theta \\ &= \int_0^\pi d\theta \int_0^1 r^3 dr \\ &= \int_0^\pi \left[\frac{1}{4} r^4 \right]_0^1 d\theta \\ &= \frac{1}{4} \int_0^\pi d\theta \\ &= \frac{\pi}{4}\end{aligned}$$

9.1.二

1. 求 $\iint_D \frac{2x}{y} d\sigma$, 式中 $D = \{(x, y) \mid y \leq x \leq 2, 1 \leq y \leq 2\}$.

解 D 是 X -型区域

$$\begin{aligned}\iint_D \frac{2x}{y} d\sigma &= \int_1^2 dx \int_1^x \frac{2x}{y} dy \\ &= \int_1^2 2x [\ln y]_1^x dx\end{aligned}$$

$$= \int_1^2 2x \ln x dx$$

$$= \int_1^2 \ln x dx^2$$

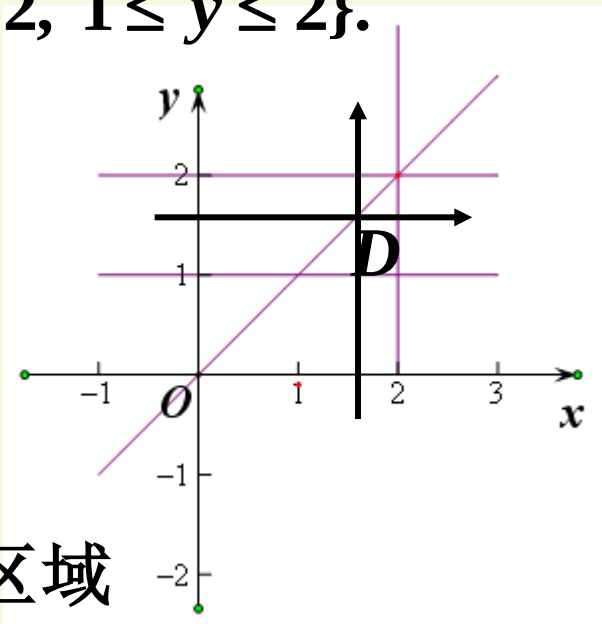
$$= [x^2 \ln x]_1^2 - \int_1^2 x dx$$

$$= 4 \ln 2 - \frac{1}{2} [x^2]_1^2$$

$$= 4 \ln 2 - \frac{3}{2}$$

D 是 Y -型区域

$$\begin{aligned}\iint_D \frac{2x}{y} d\sigma &= \int_1^2 dy \int_y^2 \frac{2x}{y} dx \\ &= \int_1^2 \frac{1}{y} [x^2]_y^2 dy = \int_1^2 \left(\frac{4}{y} - y \right) dy \\ &= [4 \ln y - \frac{1}{2} y^2]_1^2 = 4 \ln 2 - \frac{3}{2}\end{aligned}$$



9.1.二

2. 求 $\iint_D (x+y) dx dy$, 式中 $D = \{(x, y) \mid y = |x|, y = 2|x|, y = 1\}$.

解 D 是 Y-型区域

$$\iint_D (x+y) dx dy$$

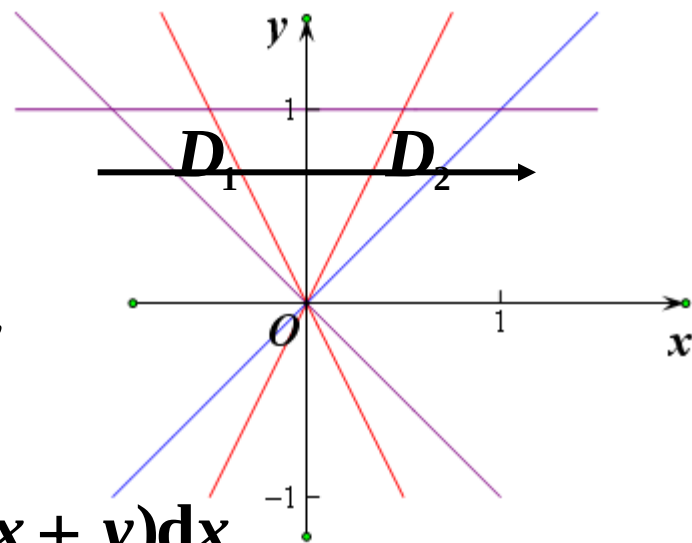
$$= \iint_{D_1} (x+y) dx dy + \iint_{D_2} (x+y) dx dy$$

$$= \int_0^1 dy \int_{-\frac{y}{2}}^{-\frac{y}{2}} (x+y) dx + \int_0^1 dy \int_{\frac{y}{2}}^{\frac{y}{2}} (x+y) dx$$

$$= \int_0^1 \left[\frac{1}{2} x^2 + xy \right]_{-\frac{y}{2}}^{-\frac{y}{2}} dy + \int_0^1 \left[\frac{1}{2} x^2 + xy \right]_{\frac{y}{2}}^{\frac{y}{2}} dy$$

$$= \frac{1}{8} \int_0^1 y^2 dy + \frac{7}{8} \int_0^1 y^2 dy = \int_0^1 y^2 dy$$

$$= \left[\frac{1}{3} y^3 \right]_0^1 = \frac{1}{3}$$



9.1.二

3.求二重积分 $\iint_D x^2 e^{-y^2} dx dy$,

式中 D 是由直线 $y = x$ 、 $y = 1$ 及 y 轴围成的区域

解 D 是 Y -型区域

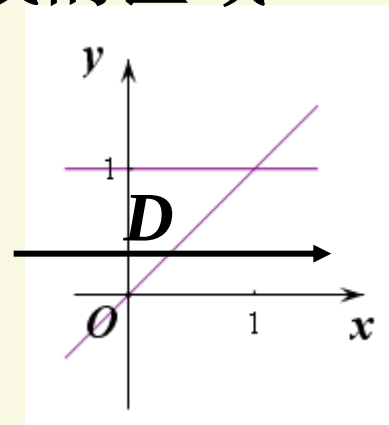
$$\iint_D x^2 e^{-y^2} dx dy = \int_0^1 dy \int_0^y x^2 e^{-y^2} dx$$

$$= \int_0^1 \frac{1}{3} y^3 e^{-y^2} dy = -\frac{1}{6} \int_0^1 y^2 de^{-y^2}$$

$$= -\frac{1}{6} [y^2 e^{-y^2}]_0^1 + \frac{1}{6} \int_0^1 e^{-y^2} dy^2$$

$$= [-\frac{1}{6} e^{-1} - \frac{1}{6} e^{-y^2}]_0^1$$

$$= \frac{1}{6} (1 - 2e^{-1})$$



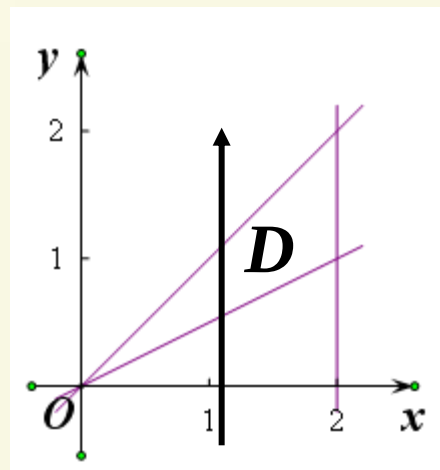
9.1.二

4. 求二重积分 $\iint_D \frac{\sin x}{x} dx dy$,

式中 D 是由直线 $y = x$ 、 $y = \frac{x}{2}$ 、 $x = 2$ 围成的区域

解 D 是 X -型区域

$$\begin{aligned}\iint_D \frac{\sin x}{x} dx dy &= \int_0^2 dx \int_{\frac{x}{2}}^x \frac{\sin x}{x} dy \\&= \int_0^2 \frac{\sin x}{x} [y]_{\frac{x}{2}}^x dx \\&= \frac{1}{2} \int_0^2 \sin x dx \\&= \frac{1}{2} [-\cos x]_0^2 \\&= \frac{1}{2} (1 - \cos 2)\end{aligned}$$



9.1.二

5. 计算 $I = \iint_D (x^2 + y^2) d\sigma$

其中 D 由 $y = x$, $y = x + a$, $y = a$ 及 $y = 3a$ ($a > 0$) 围成.

解 $D = \{(x, y) | a \leq y \leq 3a, y - a \leq x \leq y\}$

$I = \iint_D (x^2 + y^2) d\sigma$ D 是 Y -型区域

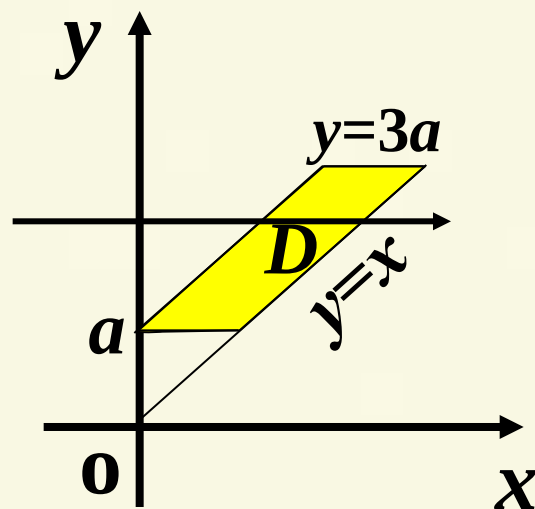
$$= \int_a^{3a} dy \int_{y-a}^y (x^2 + y^2) dx$$

$$= \int_a^{3a} \left[\frac{1}{3} x^3 + y^2 x \right]_{y-a}^y dy$$

$$= \int_a^{3a} \left[\frac{y^3 - (y-a)^3}{3} + ay^2 \right] dy$$

$$= \left[\frac{1}{12} (y^4 - (y-a)^4) + \frac{1}{3} ay^3 \right]_a^{3a}$$

$$= \frac{16}{3} a^4 + 9a^4 - \frac{1}{3} a^4 = 14a^4$$



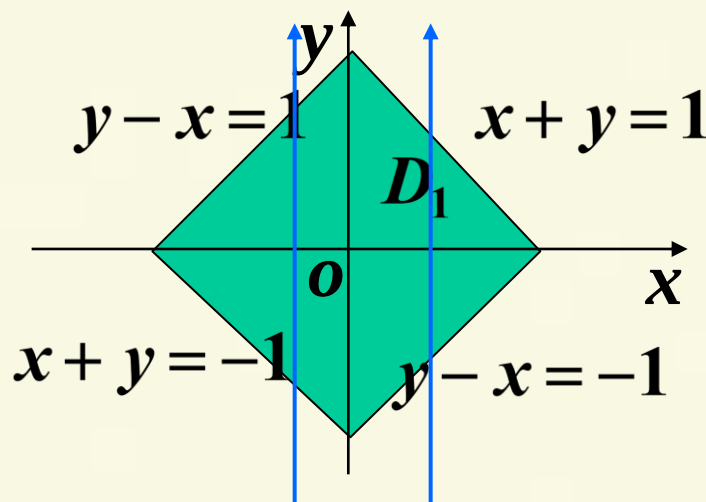
9.1.二

6. 求二重积分 $\iint_D |xy| d\sigma$, 式中 $D = \{(x, y) \mid |x| + |y| \leq 1\}$.

解: $I = \iint_D |xy| dx dy = 4 \iint_{D_1} xy dx dy$

$$= 4 \int_0^1 dx \int_0^{1-x} xy dy$$

$$= 2 \int_0^1 x(1-x)^2 dx = \frac{1}{6}$$



$$I = \int_{-1}^0 dx \int_{-1-x}^{1+x} |xy| dy + \int_0^1 dx \int_{-1+x}^{1-x} |xy| dy$$

$$= \int_{-1}^0 dx \int_{-1-x}^0 (xy) dy + \int_{-1}^0 dx \int_0^{1+x} (-xy) dy$$

$$+ \int_0^1 dx \int_{-1+x}^0 (-xy) dy + \int_0^1 dx \int_0^{1-x} (xy) dy$$

9.1.二

7. 交换积分次序 $\int_{\frac{1}{2}}^{\frac{1}{4}} dy \int_{\frac{1}{2}}^{\sqrt{y}} f(x, y) dx + \int_{\frac{1}{2}}^1 dy \int_y^{\sqrt{y}} f(x, y) dx$

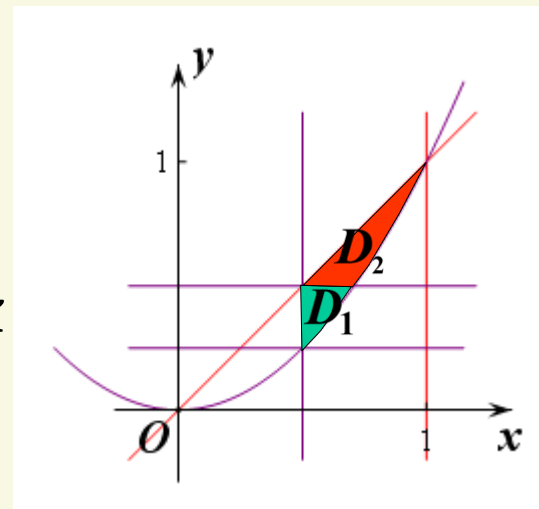
解 $D_1 = \left\{ (x, y) \mid \frac{1}{4} \leq y \leq \frac{1}{2}, \frac{1}{2} \leq x \leq \sqrt{y} \right\}$

$$D_2 = \left\{ (x, y) \mid \frac{1}{2} \leq y \leq 1, y \leq x \leq \sqrt{y} \right\}$$

$$\int_{\frac{1}{2}}^{\frac{1}{4}} dy \int_{\frac{1}{2}}^{\sqrt{y}} f(x, y) dx + \int_{\frac{1}{2}}^1 dy \int_y^{\sqrt{y}} f(x, y) dx$$

$$= \iint_{D_1 + D_2} f(x, y) dx dy$$

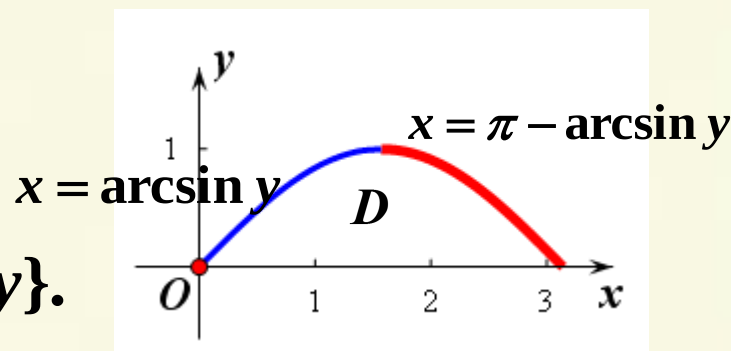
$$= \int_{\frac{1}{2}}^1 dx \int_{x^2}^x f(x, y) dy$$



9.1.二

8. 求二次积分 $\int_0^1 dy \int_{\arcsin y}^{\pi - \arcsin y} x dx$

解 $D = \{(x, y) | 0 \leq y \leq 1, \arcsin y \leq x \leq \pi - \arcsin y\}.$



$$\begin{aligned} \int_0^1 dy \int_{\arcsin y}^{\pi - \arcsin y} x dx &= \iint_D x dx dy \\ &= \int_0^\pi dx \int_0^{\sin x} x dy \\ &= \int_0^\pi x [y]_0^{\sin x} dx = \int_0^\pi x \sin x dx \\ &= [-x \cos x]_0^\pi + \int_0^\pi \cos x dx \\ &= \pi + [\sin x]_0^\pi \\ &= \pi \end{aligned}$$

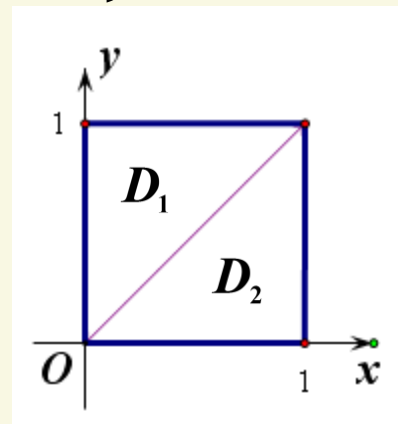
9.1.二

9. 设函数 $f(x)$ 在 $[0,1]$ 上连续, 且 $\int_0^1 f(x)dx = A$,

$$\text{求 } \int_0^1 dx \int_x^1 f(x) f(y) dy$$

解 $D_1 = \{(x, y) | 0 \leq x \leq 1, x \leq y \leq 1\}$

$$D_2 = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x\}$$



$$\text{先考察 } \int_0^1 dx \int_0^1 f(x) f(y) dy = \int_0^1 f(x) dx \int_0^1 f(y) dy$$

$$= \iint_D f(x) f(y) dx dy = \left[\int_0^1 f(y) dy \right] \int_0^1 f(x) dx$$

$$= \iint_{D_1+D_2} f(x) f(y) dx dy = \left[\int_0^1 f(x) dx \right] \int_0^1 f(x) dx$$

$$= \int_0^1 dx \int_x^1 f(x) f(y) dy = A^2$$

$$+ \int_0^1 dy \int_y^1 f(x) f(y) dx$$

$$= \int_0^1 dx \int_x^1 f(x) f(y) dy + \int_0^1 f(y) dy \int_y^1 f(x) dx$$

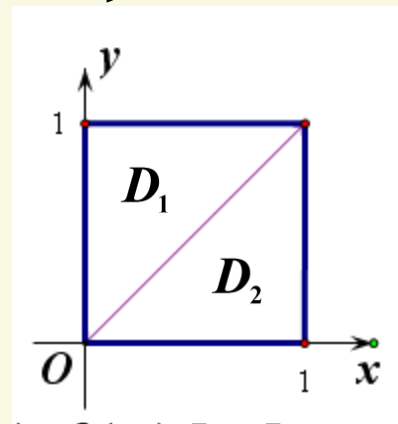
9.1.二

9. 设函数 $f(x)$ 在 $[0,1]$ 上连续, 且 $\int_0^1 f(x)dx = A$,

$$\text{求} \int_0^1 dx \int_x^1 f(x) f(y) dy$$

解 $D_1 = \{(x, y) | 0 \leq x \leq 1, x \leq y \leq 1\}$

$$D_2 = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x\}$$



$$\text{先考察} \int_0^1 dx \int_0^1 f(x) f(y) dy = A^2 = \iint_{D_1+D_2} f(x) f(y) dx dy$$

$$= \int_0^1 dx \int_x^1 f(x) f(y) dy + \int_0^1 f(y) dy \int_y^1 f(x) dx$$

$$= \int_0^1 dx \int_x^1 f(x) f(y) dy + \int_0^1 f(y) dy \int_y^1 f(t) dt$$

$$= \int_0^1 dx \int_x^1 f(x) f(y) dy + \int_0^1 f(x) dx \int_x^1 f(t) dt$$

$$= \int_0^1 dx \int_x^1 f(x) f(y) dy + \int_0^1 f(x) dx \int_x^1 f(y) dy$$

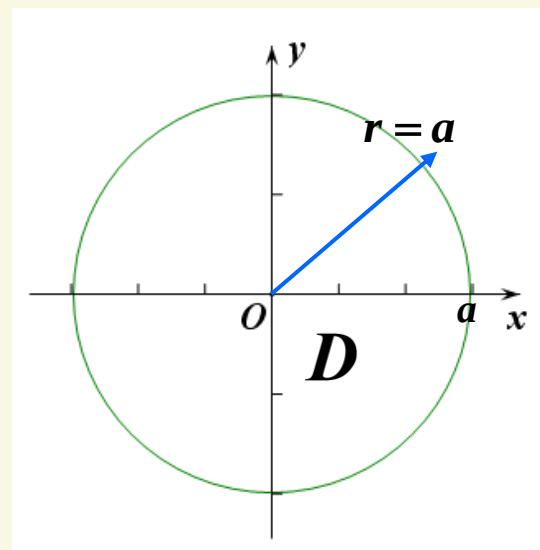
$$\text{即} \int_0^1 dx \int_x^1 f(x) f(y) dy = \frac{1}{2} A^2 = 2 \int_0^1 dx \int_x^1 f(x) f(y) dy$$

9.1.二

10. 求二重积分 $\iint_D e^{-x^2-y^2} d\sigma$, 式中 $D = \{(x, y) \mid x^2 + y^2 \leq a^2\}$.

解 利用极坐标系

$$\begin{aligned}\iint_D e^{-x^2-y^2} d\sigma &= \iint_D e^{-r^2} r dr d\theta \\&= \int_0^{2\pi} d\theta \int_0^a e^{-r^2} r dr \\&= -\frac{1}{2} \int_0^{2\pi} [e^{-r^2}]_0^a d\theta \\&= -\frac{1}{2} \int_0^{2\pi} (e^{-a^2} - 1) d\theta \\&= \pi(1 - e^{-a^2})\end{aligned}$$



9.1.二

11. 求二重积分 $\iint_D \sqrt{x^2 + y^2} d\sigma$,

式中 $D = \{(x, y) \mid 0 \leq y \leq x, x^2 + y^2 \leq 2x\}$.

解 利用极坐标系

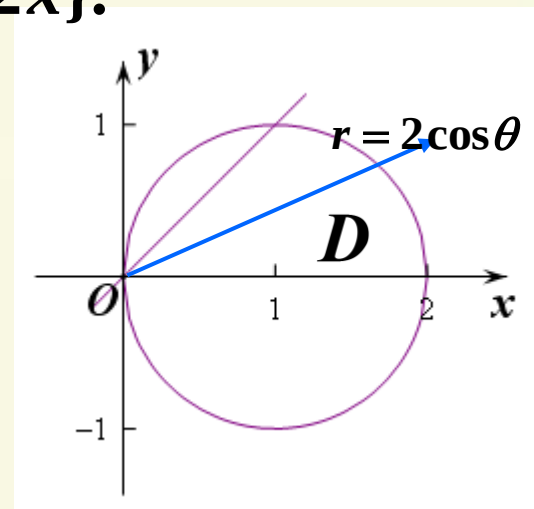
$$\iint_D \sqrt{x^2 + y^2} d\sigma = \iint_D r \cdot r dr d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_0^{2\cos\theta} r^2 dr$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{4}} [r^3]_0^{2\cos\theta} d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{4}} \cos^3 \theta d\theta = \frac{8}{3} \int_0^{\frac{\pi}{4}} (1 - \sin^2 \theta) d\sin \theta$$

$$= \frac{8}{3} \left[\sin \theta - \frac{1}{3} \sin^3 \theta \right]_0^{\frac{\pi}{4}} = \frac{8}{3} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} \right) = \frac{10\sqrt{2}}{9}$$



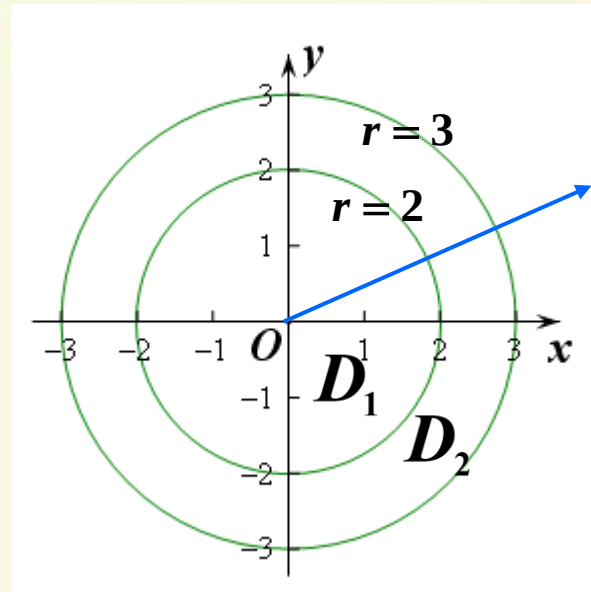
9.1.二

12. 求二重积分 $\iint_D |x^2 + y^2 - 4| d\sigma$,

式中 $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$.

解 利用极坐标系

$$\begin{aligned}\iint_D |x^2 + y^2 - 4| d\sigma &= \iint_{x^2+y^2 \leq 4} (4 - x^2 - y^2) d\sigma \\ &\quad + \iint_{4 \leq x^2+y^2 \leq 9} (x^2 + y^2 - 4) d\sigma \\ &= \iint_{D_1} (4 - r^2) \cdot r dr d\theta + \iint_{D_2} (r^2 - 4) \cdot r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 (4r - r^3) dr + \int_0^{2\pi} d\theta \int_2^3 (r^3 - 4r) dr \\ &= \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4 \right]_0^2 d\theta + \int_0^{2\pi} \left[\frac{1}{4}r^4 - 2r^2 \right]_2^3 d\theta \\ &= \int_0^{2\pi} 4 d\theta + \int_0^{2\pi} \left(\frac{81}{4} - 14 \right) d\theta = \frac{41}{2} \pi\end{aligned}$$



9.1.二

13. 求二重积分 $\iint_D \frac{x^2}{x^2 + y^2} d\sigma$,

式中 $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

解 利用极坐标系

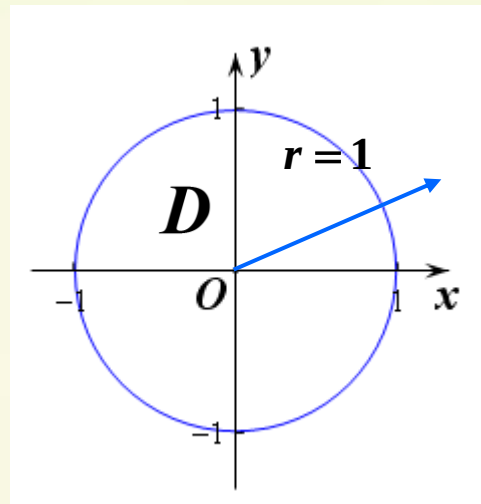
$$\iint_D \frac{x^2}{x^2 + y^2} d\sigma = \iint_{r^2 \leq 1} \frac{r^2 \cos^2 \theta}{r^2} \cdot r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r \cos^2 \theta dr$$

$$= \frac{1}{2} \int_0^{2\pi} \cos^2 \theta [r^2]_0^1 d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{1}{2} \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{\pi}{2}$$



9.1.三

证明 $\int_0^1 dy \int_0^{\sqrt{y}} e^y f(x) dx = \int_0^1 (e - e^{x^2}) f(x) dx$

证 $\int_0^1 dy \int_0^{\sqrt{y}} e^y f(x) dx$ 交换积分次序

$$= \iint_D e^y f(x) dx dy$$

$$= \int_0^1 dx \int_{x^2}^1 e^y f(x) dy$$

$$= \int_0^1 f(x) dx \int_{x^2}^1 e^y dy$$

$$= \int_0^1 f(x) [e^y]_{x^2}^1 dx$$

$$= \int_0^1 (e - e^{x^2}) f(x) dx$$

